Mathematical explanation of photonic experiments

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Basic states

Like a classical bit, a quantum bit (qubit) has two basis states. However, we use "ket" notation to represent them:

- $|0\rangle$ reprents the state 0 and
- $|1\rangle$ reprents the state 1.

NOT gate (operator)

It flips the value:

- $X|0\rangle = |1\rangle$ and
- $X|1\rangle = |0\rangle$.

Hadamard gate (operator)

It creates a combination of the basis state:

- $H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

and

- $H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$.

Here the scalar values are called as *amplitudes*.

If measured, we observe the states $|0\rangle$ and $|1\rangle$ with probability $\frac{1}{2}$ after applying a Hadamard to a basis state.

Amplitudes and probabilities

The probabilities are calculated based on amplitudes.

For example, a qubit can be in a quantum state as

$|u\rangle = a|0\rangle + b|1\rangle$.

Then, the probabilities of being in states $|0\rangle$ and $|1\rangle$ are $|a|^2$ and $|b|^2$.

As the total probability must be 1, then we have

$|a|^2 + |b|^2 = 1.$

Note that a and b can also be complex numbers.

Quantum superposition

In the above example, if both a and b are non-zero, then we say that the qubit is in a superposition (of $|0\rangle$ and $|1\rangle$).

When in a superposition, applying a quantum operator (e.g., Hadamard) can create interferences.

Re-visting the photon experiments

Experiment 1

We start in state $|0\rangle$.

We apply a Hadamard operator. Then, our quantum state is

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle.$$

We make a measurement.

Thus, we obtain $|0\rangle$ and $|1\rangle$ with probability $\frac{1}{2}$.

After the measurement, the state of qubit is the observed outcome.

Experiment 2

We repeat Experiment 1, but we continue in the case of $|1\rangle$.

We have $|0\rangle$ and $|1\rangle$ with probability $\frac{1}{2}$ after the measurement.

If the measurement outcome is $|1\rangle$, we apply a second Hadamard. Our quantum state in this branch is

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

After measuring, we obtain $|0\rangle$ and $|1\rangle$ with probability $\frac{1}{2}$.

Thus, we can observe $|1\rangle$ with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

And, we can observe $|0\rangle$ with probability $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

Experiment 3

We start in $|0\rangle$ and apply a Hadamard operator twice.

After the first Hadamard:

$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle.$$

After the second Hadamard:

$$H|+\rangle = H\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{\sqrt{2}}H|0\rangle + \frac{1}{\sqrt{2}}H|1\rangle.$$

We obtain four outcomes and we add them all:

$$\frac{1}{\sqrt{2}}\left(\left(\underbrace{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle}_{H|0\rangle}\right) + \left(\underbrace{\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle}_{H|1\rangle}\right)\right).$$

That is: $\left(\frac{1}{2}|0\rangle + \frac{1}{2}|0\rangle\right) + \left(\frac{1}{2}|1\rangle - \frac{1}{2}|1\rangle\right) = |0\rangle$.

Here $|0\rangle$s are interfered constructively, and $|1\rangle$s are interfered destructively.

Exercise

Repeat Experiment 3 by starting in $|1\rangle$.

Z Gate

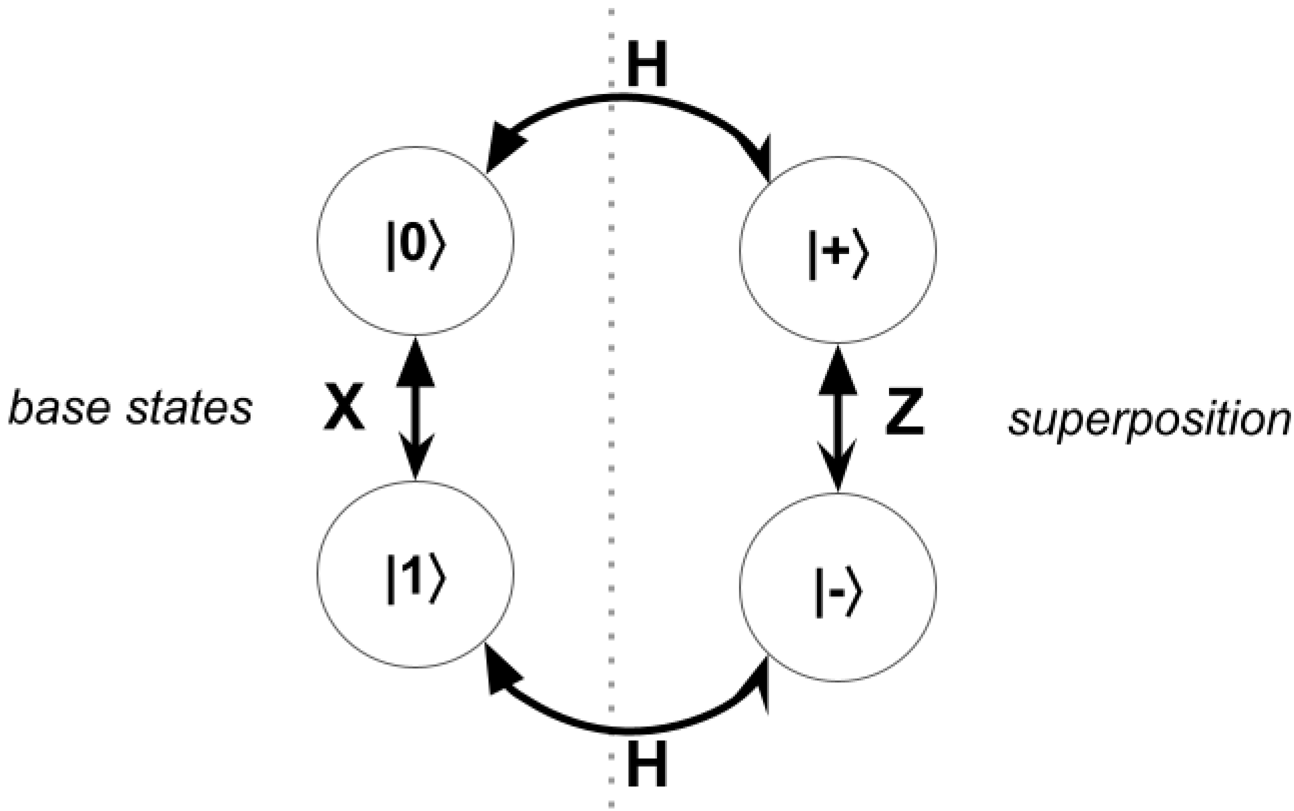
It flips the sign of $|1\rangle$.

If we are in $|+\rangle$ and apply Z, we obtain $|-\rangle$:

$$Z\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = |-\rangle.$$

Similarly, if we are in $|-\rangle$ and apply Z, we obtain $|+\rangle$.

State transition diagram



Using Matrices and Vectors

This part is optional.

We represent the quantum states as vectors.

- $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and
- $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

We represent the operators (gates) as matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix},\ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We observe that

$HH = XX = ZZ = I$.

Applying a quantum operator, say U to a quantum state, say $|u\rangle$, is represented as amtrix-vector multiplication:

$|v\rangle = U|u\rangle$, where $|v\rangle$ is the new quantum state.

Exercise

Calculate $H|0\rangle$ and $H|1\rangle$ by using matrices and vectors.